Journal of Physics: Conference Series

PAPER • **OPEN ACCESS**

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To cite this article: Yujie Wang 2022 *J. Phys.: Conf. Ser.* **2386** 012001

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Game Theory Survey: Voting Methods of Preferential Ballot

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Abstract. This paper is a survey focuses on seven basic Voting Methods for Preferential Ballots. The purpose is to give a pellucid illustration of these Methods in consideration of Game Theory, and understand their intrinsic and associative properties. In order to understand these Methods and their efficacy, multiple techniques, such as Mathematical Modeling and Computer Programming, are used for evaluation and comparison. The ultimate goal is to find a Voting Methods which elect the most democratic winner.

Keywords: The paper must have at least one keyword.

1. Introduction

Game theory is an interdisciplinary study possessing a wide scope of applications. Its intention is to produce profound comprehension and effective strategies to games, which are events that desire logical decision making. Although ancient philosophers like Plato brought about game- theatrics a long time ago, it was not until 1944 the first mathematical formulation of game theory came out[5]. Since then, game theory has been facilitating rational decision makers from all fields, such as Economy, Biology and Social Science, with the interpretation of mathematical models.

Within the field of game theory, voting is a paramount topic because the majority of modern world pursues democracy. People may hold different opinions about democracy, nonetheless equality and popularity are always among the keys. In other words, every vote matters, so the winner should be the" most favored" by" most voters".

Nowadays, most of the remarkable votes are single vote ballots, such as the United States Presidential Election. The single vote ballot simply asks the voters for their favorite candidate out of the list, and usually the candidate with the most votes will be the eventual winner. The advantage of such ballot is evident: it is the most lucid and acceptable ballot type to the public. However, the single vote ballot provides too little information about the voters' inclination in candidates to analyze the genuine preference of the populace.

In comparison, there are other ballot types more informative and comprehensive than the single vote ballot. For example, quantitative ballot demands the voters to grade the candidates according to preference. It is usually analogous to the scoring system in school, so that a voter's predilection is quantified: all grades on the candidates are between 50 to 100, while any score over 70 implies approval. To make the ballot more understandable to the voters, score intervals can be colored for auxiliary meaning, such like red represents disagreement. In such way, a quantitative ballot is enabled to collect the extent of fondness of voters to candidates in standardized quantity. The collected sufficient data provides great potential in mathematical evaluation, yet unfortunately few methods have been found

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nearly impeccable in assessing these data. More specifically, the complexity of the single vote ballot makes it hard to be understood enough by game theory.

In fact, the conundrum with respect to quantitative ballot can be narrow down by transferring the result into preferential ballot, another type of voting ballot considering the rank of candidates according to preference. By ordering scores in quantitative ballot from large to small, the candidates can be put in a rank so that the candidate with the highest score is in the first place and the one with the lowest score is at the last place. The obtained rank in preferential ballot illustrates the voter's choices of candidates with sorted numbers, which are far less complicated to deal with than the scores in quantitative ballot. In addition, the preferential ballot collects much more revealing information than single vote ballot because it takes the voters' penchant for all candidates into consideration, instead of only their favorite ones.

On account of these advantages, this paper focuses on diverse voting methods of preferential ballot in game theory. Seven primary voting methods, Plurality, Instant Runoff, Borda Count, Instant Runoff Borda, Worst Defeat, Schulze Method and Ranked Pairs, are demonstrated in this paper with their fundamental definitions and properties. Several techniques, such as mathematical models and computer programming will be used to comprehend the results of these methods in random elections. The ultimate goal of these experiments is to find a voting method as democratic as possible for the preferential ballot, while this paper considers that a condorcet method will be good enough for democracy.

2. Methods

As mentioned in Section 1, seven voting methods will be presented in this section. It is worth noting that this article doesn't exclude the condition of tied candidates, since the preferential ballot is attained from the quantitative ballot. More specifically, candidates can be in tie when they have the same score in the quantitative ballot, leading to the same rank in the preferential ballot. There are many ways to deal with ties, such as adding tie breaking votes. An elaborate algorithm in handling ties is developed by Bray in his website "WeVoteHere.org"[2]. However, this survey has no interest in ties. Thus, we do not consider about ties when defining the voting methods and illustrating their properties.

2.1. Plurality

Definition 2.1. *Plurality.*
Follow these steps to deal with a preferential ballot by Plurality.

(i) Keep only the first candidate from each vote. i.e., all we care about is the favorite candidate of each voter, so simply remove other candidates in the vote.

(ii) Count the number of votes of each candidate. The candidate with most votes is the winner.

It is evident that plurality is essentially the single vote ballot in a preferential ballot version. Although its limited use of information represents little of the voters' declination, plurality is quite comprehensible and well-accepted by the public, because it follows the sound logic of voting.

Definition 2.2. *Monotone*
If any voter ranks a candidate higher in his vote, this action will not hurt the candidate.

Intuitively, plurality is monotone. However, plurality is not democratic enough, since it doesn't tend to favor the" the most favored" candidate by the most voters. This characteristic of plurality will be explained later in Section 3.

2.2. Instant Runoff

Definition 2.3. *Instant Runoff*
After getting the preferential ballot votes, do the following steps for Instant Runoff Method.

(i) Count the votes based on their first choice candidate (plurality).

(ii) Eliminate the candidate at the last place. In other words, this candidate will be removed from all ballots.

(iii) Repeat step 1 and 2 recursively until there's only one candidate left. Then the remaining candidate is the winner of the instant runoff method.

Alternatively, the instant runoff method does plurality recursively by eliminating the last-place candidate at each round.Although the instant runoff method sounds more complicated than plurality, it does have some democratic trait which doesn't share with plurality, such as clone invariant.

Definition 2.4. *Clone Invariant*
The voting method doesn't penalize or reward a candidate with some similar candidates.

Unfortunately, the instant run-off method is not Monotone. Thus, this method is still notdemocratic enough by common sense.

2.3. Borda Count

It is important to know a tool which helps to count the votes and transfer the result into a matrix before jumping into the definition of Borda Count.

Definition 2.5. *Margin of Victory Matrix (MOV-Matrix)*
This is a matrix which displays the difference of votes between candidates. The formula of each count is as the following:

$$\begin{aligned} MOV - count = &\#of\ votes\ that\ row\ candidate\ ranks\ higher\ than\ column\ candidate \\ &- \#of\ votes\ that\ column\ candidate\ ranks\ higher\ than\ row\ candidate. \end{aligned}$$

From the definition, we can deduce that the following properties for MOV-Matrix:

- All the entries on the main diagonal are 0, since the row and column candidate are the same.
- The MOV-matrix is symmetrically inverse, i.e. $\forall i, j \in [n-1]$, $M_{ij} = -M_{ji}$. This is because the row and column candidate just switches in symmetric entries.

Definition 2.6. *the Borda Count*
After getting the preferential ballot votes, do the following steps for the Borda Count.

(i) Count the votes and summarize them into the MOV-Matrix.

(ii) Calculate the row sum of the MOV-Matrix. Then each candidate has a corresponding row sum. The candidate with the largest row sum wins by the Borda Count.

To check calculation, the sum of all the row sums is 0, since the sum of all entries of the MOV-Matrix is 0.

By definition, the Borda Count is indeed equivalent to picking the winner of the best average rank, which is ostensively reasonable to most people. Apparently, the Borda Count is a mono- tone method as ranking a candidate higher would help to increase his average rank.

However, the Borda Count has some fatal detriments of democracy, such as it is not clone invariant.

Another calamitous thing about the Borda Count is that using this method may lead us to a Nuclear War. Some reasonless voters may put their less favored but sober candidates in lower rank, which ultimately elects the craziest candidate to be the winner of the Borda Count.

Therefore, the adoption of the Borda Count on something critical may lead to catastrophic result. It is not a favorable voting method.

2.4. Instant Runoff Borda

As its name, Instant Runoff Borda is a method which combines the logic of Instant Runoff Voting and the Borda Count.

Definition 2.7. *Instant Runoff Borda*

Follow these steps to run Instant Runoff Borda Method for a preferential ballot.

(i) Organize the result of the votes by MOV Matrix and calculate its row sums.
(ii) By Borda Count, find the candidate with the lowest average rank. Delete this candidate with his corresponding column and row. Thus we get a new MOV Matrix.
(iii) Repeat Step 2 for each until there is only one candidate left.
The last remained candidate is the winner of Instant Runoff Borda.

From its definition, Instant Runoff Borda is still a comprehensible method to most people with only a little bit more complexity than the previous three methods. However, the best thing of Instant Runoff Borda is that it is a Condorcet method, which shall meet the elemental demand of democracy.

Definition 2.8. *Condorcet method*
A *Condorcet candidate is an all-win candidate, which means he wins all other candidate in a one versus one election.*
A *Condorcet method is a voting method which always elects the Condorcet candidate as its winner if there is one.*

Theorem 2.1. *Instant Runoff Borda is a Condorcet method.*

It's much fair *for* a Condorcet candidate to win in a democratic election. In contrast, failing the property *of* Condorcet is a bad sign to any rational voter.

2.5. Worst Defeat

Definition 2.9. *Worst Defeat*
Carry out the following steps to evaluate the preferential ballots by Worst Defeat.
(i) Put the result of the votes into a MOV Matrix.
(ii) For each candidate, take out their worst defeat, which is the smallest number in its row in MOV Matrix.
(iii) Compare the candidates with their worst defeat.
The candidate with the biggest worst defeat wins by Worst Defeat.

Apparently, Worst Defeat is a Condorcet Method since the Condorcet candidate will be the only one who has a positive worst defeat. It is also Monotone because ranking a candidate higher in a vote can only possibly raise its worst defeat, but not lower it.

Despite the above democratic properties, Worst Defeat bears some negative deficits. One of the disadvantages is that Worst Defeat is not fully Clone Invariant; instead, it's identical Clone Invariant. This means that a winner according to Worst Defeat might lose because similar candidates are added.

Another significant harm to democracy by Worst Defeat is that the Worst Defeat winner could lose to all the other candidates. Figure 1 is an example of when this happen.

Example 2.1. *The winner of Worst Defeat loses to all other candidates.*

	A	B	C	D	Worst Defeat
A	0	6	-8	1	-8
B	-6	0	7	1	-6
C	8	-7	0	1	-7
D	-1	-1	-1	0	-1

Figure 1. Example of Worst Defeat winner loses to all the other candidates.

In fact, these two drawbacks of worst defeat can be fixed in the later voting method.

2.6. Schulze Method

Schulze Method is a voting method similar to Worst Defeat. Actually, these two methods are the same in essential. The exact data on how much of this similarity between Schulze and Worst Defeat is still open and waits for an answer, but it is for certain that the results of the two methods align with each other for the most of time. It is safe to state that Schulze Method improves Worst Defeat by repairing some of its disadvantages.

Prior to the definition, we need to know a tool to represent the vote results other than MOV Matrix.

Definition 2.10. *Directed Weighted Graph according to Victory Matrix*
According to Victory Matrix, try to represent the results of votes by a directed weighted graph with the following rules:

- *Make each candidate be represented as a point in the graph, and their names are names of the points.*
- *For any positive entry in the MOV Matrix, draw an arrow from the row candidate to the column candidate.*
- *Label the arrow with the number from the corresponding entry in MOV-matrix. More specifically, an arrow from point A to B with a corresponding number x means that candidate A wins candidate B by x votes.*

Notice that this is a directed weighted graph in graph theory, with points of candidates as nodes, the arrow as a directed edge (arc), and the number labeled on the arrow as the edge's weight.

There are some other notions in the graph which should be understood.

Definition 2.11. *Chain and Strength*
A chain from a candidate A to another candidate B is a path from A to B directed by arrows. Be mindful that there might be more than one chain from one candidate to another. The strength of a chain is the smallest of the labeled numbers on the arrows along its way.

With the above definitions, one can understand the basic idea of Schulze Method.

Definition 2.12. *Schulze Method*
Execute the following steps to apply Schulze Method to a preferential ballot.

(i) Summarize the votes into the MOV Matrix. Draw the directed weighted graph according to the Marin of Victory Matrix.

(ii) Compare each pair of candidates with their strongest chain. In other words, for a pair of candidates A and B, find the chain with the largest strength from A to B and the one from B to A respectively.

(iii) The stronger strength implies that the candidate at the beginning of the chain chain beats the candidate at the end of the chain. The notation of candidate A chain beats B is A >> B
The candidate which chain beats all the other candidates is the winner of Schulze Method.

The following theorem verifies the validity of Schulze Method's definition.

Theorem 2.2. *There exists a unique candidate in Schulze Method that beats all other candidates. The proof of the theorem can be seen in Professor Bray's unpublished book[3].*

Schulze Method is a legitimate voting method with many positive properties, such as begin Condorcet, Clone Invariant and Monotone. Though it's a little bit too complicated for non-mathematician to accept, it is a sounded voting method for a democratic vote.

However, Schulze Method sometimes can behave out of normal: the Top 3 winners of Schulze Method may not be the winner by Instant Runoff.

2.7. Ranked Pairs

Definition 2.13. *Ranked Pairs(Game Theory)*
The following steps deal a preferential ballot with Ranked Pairs Method.

(i) Count the votes into the Margin of Victory Matrix. Draw the directed weighted graph according to the Margin of Victory Matrix.

(ii) List directed edges according to their weights from large to small.

(iii) Build an inequality chain for the candidates with the following rules:

- *The candidate as the tail of a directed edge is greater than the whom as the head of the edge.*
- *From the largest to the smallest, for each arc try to include its entailed inequality to an unified chain with respect to inequality.*
- *If a directed edge gives an inequalit which contradicts the previously constructed inequality chain, eliminate such arc.*

The resulted inequality chain implies the order of candidates according to Ranked Paris Method. A > B means candidate A beats candidate B.

From the definition, it is evident that one can obtain a conclusive inequality chain. Here is an example to illustrate how Ranked Pairs Method works.

Example 2.2. *Proceed Example with Ranked Pairs*

It's worth noting that Ranked Pairs has another definition with respect to Lp norm.

Definition 2.14. *Lp norm*

A sub *The L^p* norm *of a set S is the sum of the p−power to each element of S, i.e.,*

$$||\{x_k\}_{k=1}^n||_{L^p}^2 = \sum_{k=1}^{n} |x_k|^p$$

k=1

, with $p > 0$, $x_k \geq 0$.

Definition 2.15. *L^∞ ordering*

If $x_k{}_{k=1}^n > y_l{}_{l=1}^m$ is L^∞ ordering, then there exists a sufficiently large $P > 0$, such that for all $p > P$,

$$||\{x_k\}_{k=1}^n||_{L^p}^2 > ||\{y_l\}_{l=1}^n||_{L^p}^2$$

Here are two simple examples of the definitions:

Example 2.3. *L^∞ ordering*

- {9, 3, 2} > {7, 6, 5, 5, 2, 2, 2, 2}
- {9, 7, 7, 6, 5} > {9, 7, 7, 6, 3, 2, 2, 2, 1}

Comment: By definition, we can simply compare two sets' L∞ ordering by comparing their largest elements. In the first example, 9 > 7; while in the second one, 9, 7, 7, 6 are the same largest elements in both sets, and 5 > 3.

Recall that each candidate represents a list in the Margin of Victory (MOV) matrix. Therefore, view each candidate as a list, we can define Ranked Pairs of L∞ ordering.

Definition 2.16. *Ranked Pairs Method of L ∞ ordering*

Ranked Pairs chooses the ordering of the candidates so that, when the Margin of Victory Matrix is written in the Ranked Pair order, the Lp norm of the negative entries in the upper triangular part of the MOV matrix is minimized, for all p > P .

In order to understand the above definition, we demonstrate Figure 2 as a defined step of Ranked Pairs based on L∞ ordering.

In order to understand the above definition, we demonstrate Figure 2 as a defined step of Ranked Pairs based on L∞ ordering.

Example 2.4. *Ranked Pairs Method of L∞ ordering In the MOV matrix, candidate B is removed*

	C	D	A	B
C	0	4	7	-3
D	-4	0	5	6
A	-7	-5	0	2
B	3	-6	-2	0

Figure 2. Example of Ranked Pairs Method of L∞ ordering.

as the last place of Ranked Pairs because:

$$||(C, D, A)||^2_{L^p} + 3^p \leq ||(suborder)||^{RP}_{L^p} + 3^p$$
$$||C, D, A, B||^2_{L^p} \leq ||\forall\ order\ with\ B\ in\ last\ place||^{RP}_{L^p}$$

Therefore, we eliminate candidate B in this round, because it is the last place of the Ranked Pair Method among the four candidates.

With these two definitions of Ranked Pairs, several theorems can be proved in both interpretation of the methods.

Theorem 2.3. *Ranked Pairs Method gives a meaningful ordering of candidates.*
Proof. The theorem can be translated into the following claims:

If candidate A is one place before candidate B in the Ranked Pair ordering, then candidate A must beats candidate B.

This claim is evident conforming to the above definition if Ranked Pairs Method.

By the second definition which is of L∞ ordering, suppose that there exists candidate C such that it's one place before candidate D in the Ranked Pair ordering, but it's L∞ ordering is smaller than candidate D. Then the MOV Matrix in the upper triangular part will be smaller with candidate D instead of C, which contradicts to the Ranked Pairs definition of L∞ ordering.

There are some more theorems explain the positive characteristics of Ranked Pairs.

Theorem 2.4. *Ranked Pairs Method is of Stability, i.e. removing the first or last few candidates will not affect the resulting order of Ranked pairs.*

Theorem 2.5. *In Ranked Pairs, If we have three set A,B,C, then if according to the L-infinity ordering A > B and B > C, then A > C.*

3. Techniques

In order to assess the performance of these seven voting methods, a few techniques are demanded to generate a meaningful background for this assessment, including using computer programming to produce random elections with mathematical models.

3.1. Mathematical Models

Construction of a mathematical model will be the first step for mathematicians to transfer a real world problem into a mathematical one. This paper utilizes a geometrical interpretation for the preferential ballot, which shall be visually coherent.

Here we introduce three types of models, which can be extended to a more general version of geometric mathematical models for preferential ballots.

(i) the Unit Interval

The voters are distributed uniformly across a unit interval from [0,1], while the candidates are located on the same interval according to their stands. For example, in a political vote, we can view 0 as the left and 1 as the right. In fact, most of the voters favor the center, which is at 12 .

(ii) the Unit Equilateral Triangle

The voters are distributed uniformly across an unit equilateral triangle, while the candidates are located on the same geometry according to their stands.

(iii) the Unit Square Model

The voters are distributed uniformly across a unit square, while the candidates are located on the same unit square according to their stands.

In short, the location of the voter represents its preference to each candidate in accordance to the distance in between. The closer the voter is to the candidate, the more he prefers this particular person. It is worth noting that the Unit Square Model has a special property, which makes it less interesting than the Unit Equilateral triangle.

Theorem 3.1. *The Unit Square Model is of point symmetry.*
This is saying that for any point in the Unit Square Model, there is always another point which shares the same distance from the central point, and any line through the middle point divides the unit square into two symmetric areas.
The point symmetry can be observed in Figure 3.

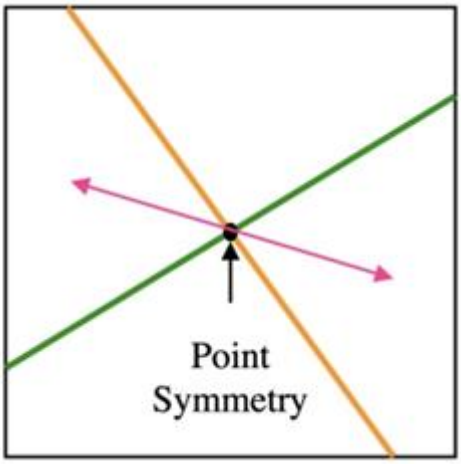

Figure 3. Point Symmetry of the Unit Square Model.

Thence, we can utilize the point symmetry to prove that the Unit Square Model favors the central.

Theorem 3.2. *In a Unit Square Model, the candidate closest to the center beats all other candidates.*
This theorem can be proved with the following algorithm:

(i) For any two points A,B in the unit square, connect them with line segment AB.
(ii) Draw the perpendicular bisector l1 of line segment AB.
(iii) Draw a line l2 parallel to l1 through the center.

From Figure 4, we can observe that there is a half of the unit square contains in the area belongs to the more central candidate.

As a consequence, this paper prefers to evaluate the voting methods with non-point symmetric models, such as the Unit Triangle Model or other odd-sided equilateral polygons. Irregular geometries can also be created by the later computer program for studying.

3.2. Random Elections

Developing a random election helps to remove bias in our evaluation. In particular, a randomized election allows us to compare the voting methods in a disinterested environment, with these arbitrarily produced votes. Here are several common types of random elections.

- *"Maximally Random". "Maximally Random" generates arbitrary preferential votes and see who wins according to each method.*
- *Random MOV Matrix. The Random MOV generates random numbers for the entries of the MOV Matrix by obeying the basic rules of MOV-Matrix.*

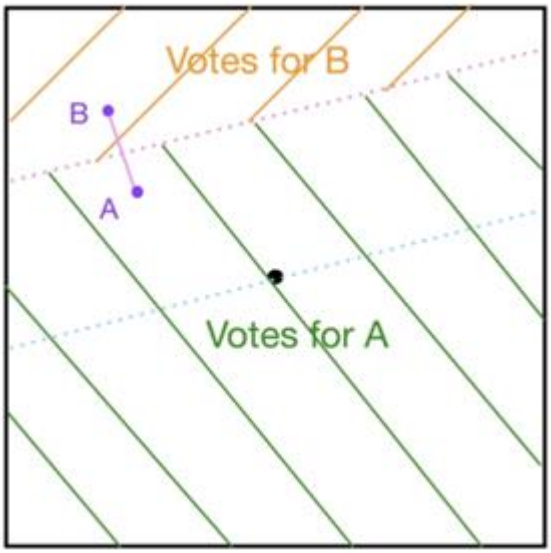

Figure 4. The Unit Square Model always favor the center.

In fact, the Random MOV is even more random than the "Maximally Random".

The random elections bring voting results by chance and eliminates the potential experimental bias, which is critical to our later examination on different voting methods.

3.3. Computer Programming

Computer science has achieved incredible progress in the past few decades, which to some extent impact mathematicians' approach to problems. This paper borrows the codes from Professor Bray. Professor Bray produces a marvelous computer program in Matlab, which displays the results of voting methods on the same geometric model and enables us to compare their performance.

(i) Bray's codes

Professor Bray's computer programming implement multiple tasks to serve the ultimate goal of comparing the performance of voting methods in random selection and. Here are three methods we need to display the results with Bray's codes.

- *WinningSpots4*

This method compares all voting methods on the same geometric model and displays them in the order of: Plurality, Instant Runoff, Borda Count, Instant Runoff Borda, Worst Defeat, Instant Runoff Worst Defeat, Schulze, Ranked Pairs, and Condorcet Instant Runoff Voting.

The voters uniformly distribute across the model. The numbers on the model represent where the candidates locate by their stands. The colors show where another candidate should go to change the election results. If the additional candidate stands in the green area, it means that he will win the election; in the area of yellow, the candidate still wins the election according to the

voting method, but there's some candidate who beats him; in the area of orange, the candidate doesn't win the election but changes the winner; in the area of red, the candidate beats the winner but doesn't change the outcome of election.
The result of this method is presented in Figure 5.

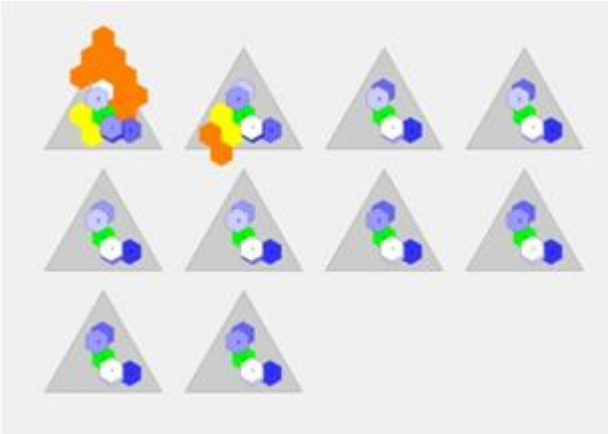

Figure 5. WinningSpots4([0 1 2]*360/3, [1 1 1], .5, 5, 1.2, 20, [1 2 3 4 5 6 7 8 9 10]).

• *AllMethods*

This method analyzes Ranked Pair on the geometric model, by showing the area of supporters for each candidate. On the top graph of Figure 6, some labels are beside the candidate number, indicating that this particular candidate wins by this voting methods. In this way, it is naturally reasonable to check if a voting method is democratic by the result that if the voter among one of the closest to the center of the geometry wins.

The later figures are processed by Instant Runoff Method, so that for each round the candidate with the smallest area will be eliminated, and the colors will be redistributed. The result of this method after the first round is presented in the bottom graph of Figure 6.

(ii) Python Test

Another short Python program was built for application. This Python program helps to transfer the votes into a MOV matrix, so that a Condorcet candidate can be easily detected. It is needed for a later survey. Though better interface such as Professor's Website "WeVoteHere"[2] is available, this Python test is still a good practice for starter.

Besides the triangle, Bray's codes can generate all kinds of polygons for models. In this paper, we are not going too deep for programming to see its principal and algorithms. But at least we know some of its functioning so that we can utilize its results for math research.

4. Observations

For now we have all the techniques we needed. After rounds of experiments, some observations can be concluded from the results.

4.1. Plurality and Instant Runoff

From Professor Bray's codes, it is evident to see that Plurality and Instant Runoff do not prefer the center for the most of the time. Their winner is very far off the model's center. Therefore, they are not democratic enough from this paper's view, which is introduced at the first part. Here's a short proof about Plurality, showing that this most employed voting method favors the sides more than the middle in a Unit Interval Model.

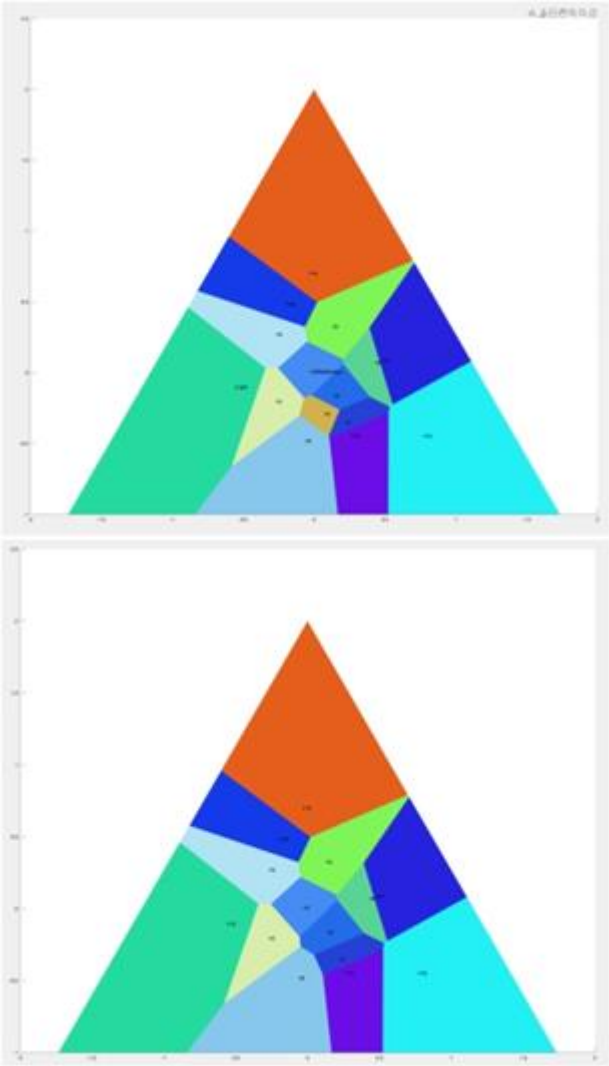

Figure 6. AllMethods([0 1 2]*360/3, [1 1 1], .5, 15).

Theorem 4.1. *According to Plurality, if there are already two candidates A and B locate on* $\frac{1}{4}$ *and* $\frac{3}{4}$ *already, there is nowhere a third candidate C can go to win.*

Proof. There are 3 cases for candidate *C* to go on point *x* in the Unit Interval.

(Case I) If $x \in [0, \frac{1}{4})$. Then B wins with a half interval.

(Case II) If $x \in [\frac{3}{4}, 1]$. This is a symmetric case to Case I, that *A* wins with a half interval.

(Case III) If $x \in [\frac{1}{4}, \frac{3}{4}]$. Since *A, B* are symmetric to each other from the midpoint of the interval, without loss of generality, suppose *C* is closer to *A* than *B*, i.e. $x \in [\frac{1}{4}, \frac{1}{2})$

Then *B* gets:

$$\begin{aligned}
&1-(\frac{3}{4}+x)\frac{1}{2}\\
=&1-\frac{3}{8}-\frac{1}{2}x\\
=&\frac{5}{8}-\frac{1}{2}x\\
\geq&\frac{5}{8}-\frac{1}{2}\cdot\frac{1}{2}\\
=&\frac{3}{8}
\end{aligned}$$

And C gets $\frac{1}{4}$ votes. Thus, B gets more votes than C. Therefore, C can never win.

Therefore, C can never win.

For a Unit Interval Model, the center at $\frac{1}{2}$ should be favored by all voters, which should be our winner by democracy. This theorem to some extent explains the two-party system in the U.S., which posts a question whether Plurality is a democratic voting method to elect a "most favored" candidate.

4.2. Comparison among Worst Defeat, Schulze and Ranked Pairs

For these three most skilled methods in electing the most centered candidate, they are essentially connected to each other. As mentioned in Part 2, Schulze is an improvement of Worst Defeat. Though it is not verified, some researcher does claim that these two methods share 99.9% same results. In the meantime, Schulze and Ranked Pairs are comparable with 99% agreement in results.

Theorem 4.2. *With three or fewer candidates, Worst Defeat, Schulze and Ranked Pairs always get the same results.*

Proof. The above statement can be easily verified by enumeration.

However, Ranked Pairs is considered as the best of the three, because it gives a meaningful rank of the candidates, which can be useful in the real world's application.

4.3. Condorcet Methods

From the results of Bray's computer program, we observe that Condorcet Methods implements an exceptional job in picking a winner close to the center. Although Schulze and Ranked Pairs seem to be the methods have the most of the positive properties, all Condorcet Methods actually are adequate for a democratic election. Some expertise in Game Theory even claim Condorcet Methods agree for 90% in a computer randomized preferential ballot, and even 99% of the time in the real world. This is because in any election in the real world, a Condorcet Winner is highly likely to appear.

In order to verify this, two preferential ballots are created. One of them is to ask 15 voters for their preference in 15 flavors of ice-cream. The Python code helps to process the votes, and from its generated MOV Matrix in Figure 7, a condorcet winner, vanilla appears. A condorcet winner also shows up in another survey on "which city you like the most". Another interesting corollary from the definition of Condorcet Methods is that if there exists a Condorcet Winner, he can always beat a winner from a non-condorcet method. Which means we can always win any supporter for a non-condorcet method by a one-versus-one candidates head up competition.

Therefore, any Condorcet Method will be good enough in a real-world standard.

```
A1 = [3, 4, 2, 15,2, 6, 6, 4, 5, 1, 5, 9, 14, 1, 2]
B1 = [5, 13, 6, 10, 10, 4, 15, 8, 11, 7, 13, 13, 15, 14, 7]
C1 =[1, 1, 7, 1, 1, 1, 5, 2, 1, 3, 2, 2, 1, 3, 3]
D1 = [6, 10, 10, 5, 5, 9, 12, 7, 6, 2, 14, 10, 9, 15, 8]
E1 = [4, 12, 15, 7, 15, 15, 13, 15, 10, 14, 15, 3, 3, 5, 15]
F1 = [2, 3, 5, 3, 11, 5, 14, 9, 8, 15, 4, 15, 4, 9, 12]
G1 = [9, 14, 3, 2, 6, 12, 10, 13, 9, 8, 10, 5, 7, 6, 11]
H1 = [10, 5, 14, 13, 4, 7, 1, 5, 15, 10, 9, 14, 10, 2, 9]
I1 = [14, 6, 12, 11, 8, 8, 2, 6, 13, 9, 8, 12, 11, 13, 10]
J1 = [15, 2, 4, 4, 9, 10, 4, 1, 4, 4, 1, 1, 2, 4, 1]
K1 = [12, 8, 13, 12, 13, 11, 7, 11, 3, 11, 11, 6, 13, 11, 14]
L1 = [11, 7, 9, 14, 3, 3, 3, 3, 14, 12, 6, 7, 12, 10, 4]
M1 = [13, 11, 1, 9, 12, 2, 9, 10, 2, 5, 7, 8, 8, 12, 5]
N1 = [7, 9, 11, 6, 14, 13, 11, 12, 12, 13, 3, 4, 5, 7, 13]
O1 = [8, 15, 8, 8, 7, 14, 8, 14, 7, 6, 12, 11, 6, 8, 6]

[0, 11, -7, 11, 9, 3, 9, 9, 9, -3, 7, 3, 3, 7, 11]
[-11, 0, -13, -3, -1, -3, -3, 1, -1, -11, 1, -5, -9, 1, -5]
[7, 13, 0, 13, 15, 13, 13, 11, 13, 3, 15, 13, 13, 15, 15]
[-11, 3, -13, 0, 7, -1, 1, 1, 3, -7, 3, -5, -3, 3, 3]
[-9, 1, -15, -7, 0, -5, -5, -5, -3, -13, -5, -3, -5, -5, -3]
[-3, 3, -13, 1, 5, 0, -1, 1, 3, -9, 7, 1, 1, 5, 1]
[-9, 3, -13, -1, 5, 1, 0, -1, 3, -7, 5, 1, -3, 3, 3]
[-9, -1, -11, -1, 5, -1, 1, 0, 3, -5, 7, -1, -3, 1, -1]
[-9, 1, -13, -3, 3, -3, -3, -3, 0, -7, 7, -3, -7, -1, -5]
[3, 11, -3, 7, 13, 9, 7, 5, 7, 0, 11, 7, 7, 13, 11]
[-7, -1, -15, -3, 5, -7, -5, -7, -7, -11, 0, -7, -5, -1, -1]
[-3, 5, -13, 5, 3, -1, -1, 1, 3, -7, 7, 0, 3, 1, 1]
[-3, 9, -13, 3, 5, -1, 3, 3, 7, -7, 5, -3, 0, 1, 3]
[-7, -1, -15, -3, 5, -5, -3, -1, 1, -13, 1, -1, -1, 0, 3]
[-11, 5, -15, -3, 3, -1, -3, 1, 5, -11, 1, -1, -3, -3, 0]
```

Figure 7. Icecream Survey with Condorcet Winner.

4.4. Open Question

From Theorem 4.1, a problem naturally follows: does Plurality behave the same in other models in different polygons? That is, is there always a condition satisfied in a polygon model, that one additional candidate can never win by Plurality? For example, in an equilateral unit triangle, can three candidates prevent a fourth candidate from winning? If so, in the same model, can two candidates prevent a third one from winning?

This question can be extended in many dimensions. One good kickoff is to start from the following question:

> Find a geometry that two candidates cannot prevent the third candidate to win by Plurality.

As for some ideas to explore the problem, computer programming is definitely an efficient tool to help in some geometry. However, a rigorous proof in geometry may be needed. What's more, more complicated geometries, such as those in multi-dimensional, algebraic or topological space, can also be considered as certain cases in this question. There are probably certain classes of geometries satisfy the above requirement.

This is an open question that no one has an answer yet. To make it more formalized, this paper posts a conjecture in a more generalized version of this question:

> Conjecture of Plurality
> Plurality is a voting method such that a certain number of candidates can prevent one additional candidate to win in any model of geometry.

This conjecture is waiting for further research and answer.

5. Conclusion

Since Game Theory is such an applied math topic, it is almost impossible to neglect the real world situation in its discussion. So what is the democratic voting method we are looking for? As answered previously, this paper believes that a Condorcet Method is capable of a real world election. While in the

real world, one of the most significant properties of a good voting method is to be understandable, so that most people can accept its idea and agree with its result. As a matter of fact, the importance of Condorcet Property has been widely recognized in previous researches. In short, a comprehensible voting method with considerable Condorcet Efficiency [4] should be considered sufficient for democracy.

As an applied branch of Mathematics, the researches in Game Theory will certainly rely on mathematical modeling. Though many elections have graphics to present their results statistically, geometry has not been extensively employed in this area. Therefore, an prediction to the future of this subject is that Mathematicians are not restrained to only analytical and statistical study but utilize many other mathematical tools to understand the problem. Conversely, Game Theory may prompt new challenge in other field of Math, such as the open question proposed previously. What's more, the researches interrelate voting methods and democracy will keep evolving. Explicit aspect of democracy, such as the two-party system [1], will be scrutinized in accordance with researchers' interests in different political system.

Last but not least, the consideration of needs is necessary for different applications. For example, Ranked Pairs should be great for a basketball team election, since it can give a rank of the players. If the ballot is for a survey for cancer treatment, the Borda Count may elect a radical prescription, which may eventually threat the patient's life.

In conclusion, the voting methods in preferential ballot is such a revival topic continuously inspiring mathematicians and social scientists.

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